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Fixing Subgraphs

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Suppose G is a graph without loops or digons and H is a spanning subgraph of G. Let $A(G)$ be the automorphism group of G. The subgraph H belongs to the set $\mathscr{F}(G)$ of *fixing subgraphs of* G if and only if G contains exactly $|A(G)|/|A(H)|$ subgraphs isomorphic to H. Clearly $G \in \mathscr{F}(G)$. This paper considers: (i) basic properties of $\mathscr{F}(G)$; (ii) minimal members of $\mathscr{F}(G)$; and (iii) G such that $|\mathscr{F}(G)| = 1$.

INTRODUCTION

In this paper we introduce the concept of a fixing subgraph of a graph. For convenience we assume that all graphs are finite, without loops or digons. Let G be a graph. Let H be a spanning subgraph of G. The subgraph H belongs to the set $\mathscr{F}(G)$ of *fixing subgraphs* of G if and only if G contains exactly $|A(G)|/|A(H)|$ subgraphs isomorphic to H. Clearly $G \in \mathscr{F}(G)$. This concept, it is hoped, will shed a little light on the relationships between the symmetries of a subgraph of a graph and the graph itself and has some relevance to Ulam's famous conjecture [2, 9].

The idea of a fixing subgraph is by no means a simple one and therefore some space is given to clarifying this idea.

This paper considers: (1) basic properties of $\mathscr{F}(G)$; (2) minimal members of $\mathscr{F}(G)$, and (3) G such that $|\mathscr{F}(G)| = 1$. Define G to be *stable* when $A(G - \lambda) \subseteq A(G)$ for some $\lambda \in E(G)$, so that $|\mathscr{F}(G)| = 1$ when G is not stable. Then the main theorems are:

THEOREM 7. *If G is one of Tutte's cages* [7] *there exists a forest $F \in \mathscr{F}(G)$.*

THEOREM 10. *T is a tree with $|\mathscr{F}(T)| = 1$ if and only if T is trivial, T is an arc of length ≥ 2, or T is isomorphic to one of the graphs in Fig.* 1:

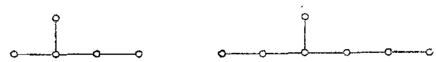
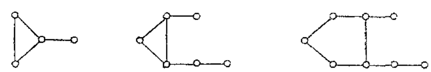


FIGURE 1

THEOREM 11. *A connected monocyclic graph M is unstable if and only if M is isomorphic to one of the graphs in Fig.* 2:



and the infinite family:

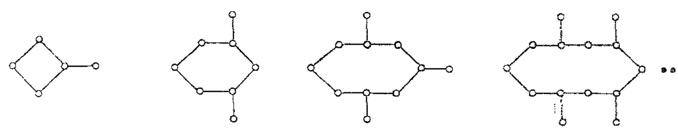


FIGURE 2

1. Let G be a finite graph without loops or digons. Let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectively. Let $a, b \in V(G)$. $[a, b]$ denotes an edge which has end vertices a and b. Assume $|V(G)| = n$, where n is some positive integer. U is a *spanning subgraph* of G if U is a subgraph of G and $V(U) = V(G)$. Let $A(G)$ denote the automorphism group of G. Let $\lambda \in E(G)$. Let $G - \lambda$ denote the subgraph of G defined by $V(G - \lambda) = V(G)$, $E(G - \lambda) = E(G) - \{\lambda\}$. G is *stable* if there exists $\lambda \in E(G)$ such that $A(G - \lambda) \subseteq A(G)$ and G is *unstable* otherwise. G is *completely stable* if, for all $\lambda \in E(G)$, $A(G - \lambda) \subseteq A(G)$. Subgraphs U_1 and U_2 of G are *similar* if there exists an automorphism of G which sends U_1 into U_2.

MAIN DEFINITION. Let G be a graph and U a spanning subgraph of G. Then U is a *fixing subgraph of* G if G contains exactly $|A(G)|/|A(U)|$ subgraphs isomorphic to U. Let $\mathscr{F}(G)$ be the set of fixing subgraphs of G.

THEOREM 1. *If $U \in \mathscr{F}(G)$ then $A(U) \subseteq A(G)$.*

Proof. Let G contain k mutually dissimilar subgraphs which are isomorphic to U; then, since $U \in \mathscr{F}(G)$, we have

$$|A(G)|/|A(U)| = k\,|A(G)|/|A(G) \cap A(U)|\,.$$

Hence, since $k \geqslant 1$, $A(U) \subseteq A(G)$.

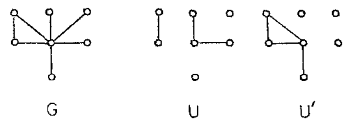


FIGURE 3

EXAMPLE 1. Let G, U, U' be as in Fig. 3. By inspection $A(U) \subseteq A(G)$, $A(U') \subseteq A(G)$, $|A(G)|/|A(U)| = 6$, and $|A(G)|/|A(U')| = 4$. It is easy to check that G contains exactly 6 subgraphs isomorphic to U and 4 subgraphs isomorphic to U'. Therefore, $U, U' \in \mathscr{F}(G)$, Of course it is usually exceptionally difficult to count the number of subgraphs isomorphic to a given subgraph (see Sections 2, 3, and 4 for special cases in which this difficulty can be partly avoided).

Equivalent Characterizations

We now state (Theorems 2 and 3) characterizations of fixing subgraphs which we believe to be both useful and intuitively enlightening. Let U be a spanning subgraph of G. Denote by $\mathscr{C}_G(U)$ the set of graph injections $\alpha : U \to G$, given by permutations of $V(G)$. Then $\mathscr{C}_G(G) = A(G)$. We state the next theorem leaving the proof as an exercise for the reader.

THEOREM 2. *Let U be a spanning subgraph of G. Then $U \in \mathscr{F}(G)$ if and only if $A(G) = \mathscr{C}_G(U)$.*

Proof. Follows immediately from the definition and Theorem 1.

Before stating Theorem 3 we require some further definitions. Let $S = \{1, 2,..., n\}$. Let σ be an injection of $V(G)$ into S. Then $G(\sigma)$ denotes the ordered pair (G, σ). σ is a *labeling* of G. We write $G(\sigma) \equiv G(\sigma')$ if and only if there exists an isomorphism μ of G into G such that $v\sigma = (v\mu)\,\sigma'$ for each $v \in V(G)$. Clearly this is an equivalence relation and the number k of equivalence classes is equal to $n!/|A(G)|$. Let

$$\pi(G) = \{G(\sigma_1), G(\sigma_2),..., G(\sigma_k)\}$$

be a set of class representatives. Let $r_i(s, s')$ be the number of edges in $G(\sigma_i)$ which have end-vertices labeled s and s' ($r_i(s, s') = 0$ or 1).

Finally let G_{ij} be the graph defined by : (i) $V(G_{ij}) = S$; (ii) there exists exactly one edge with end-vertices s and s' if $r_i(s, s') + r_j(s, s') = 2$, $s, s' \in S$. Otherwise s and s' are not adjacent. G_{ij} is called a *G-intersection graph*. Let int(G) be the set of G-intersection graphs.

THEOREM 3. *Let U be a spanning subgraph of G. Then $U \in \mathscr{F}(G)$ if and* *only if G is the only element of* $\operatorname{int}(G)$ *which contains a subgraph isomorphic to U.*

Proof. This follows immediately from the definitions and Theorem 2.

Remark 1. Loosely speaking, Theorem 3 states that a spanning subgraph U of G is a fixing subgraph of G if it has a unique extension to G.

Elementary Properties

THEOREM 4. *Let U and K be spanning subgraphs of the graph G. Suppose $U \subseteq K$. Then if $U \in \mathscr{F}(G)$, $K \in \mathscr{F}(G)$.*

Proof. Follows immediately from the definition of fixing subgraphs.

Remark 2. Notice that we have:

(i) $U \subseteq K \subseteq G$, $U \in \mathscr{F}(G) \Rightarrow K \in \mathscr{F}(G)$.

This is the statement of Theorem 4.

(ii) $U \subseteq K \subseteq G$, $U \in \mathscr{F}(G) \nRightarrow U \in \mathscr{F}(K)$.

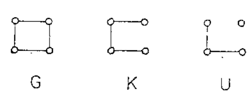


FIGURE 4

For example, consider the graphs in Fig. 4. Then $U \subseteq K \subseteq G$ and $U \in \mathscr{F}(G)$. However, $A(U) \nsubseteq A(K)$, so, by Theorem 1, $U \notin \mathscr{F}(K)$.

On the other hand we have:

THEOREM 5. $U \subseteq K \subseteq G$, $A(K) = A(G)$, $U \in \mathscr{F}(G) \Rightarrow U \in \mathscr{F}(K)$.

Proof. Assume $U \subseteq K \subseteq G$, $A(K) = A(G)$ and $U \in \mathscr{F}(G)$. By Theorem 1, $A(U) \subseteq A(G)$. Therefore $A(U) \subseteq A(G) = A(K)$. Therefore K contains exactly $|A(K)|/|A(U)|$ subgraphs similar to U. Suppose K contains m subgraphs isomorphic to U and $m > |A(K)|/|A(U)| = |A(G)|/|A(U)|$. Then, since $U \subseteq K$, G contains at least m subgraphs isomorphic to U. Since $U \in \mathscr{F}(G)$, $m \leqslant |A(G)|/|A(U)|$. This is a contradiction. Therefore K contains exactly $|A(K)|/|A(U)|$ subgraphs isomorphic to U. Therefore $U \in \mathscr{F}(K)$.

(iii) $U \subseteq K \subseteq G$, $U \in \mathscr{F}(K)$, $K \in \mathscr{F}(G) \nRightarrow U \in \mathscr{F}(G)$.

For example, consider the graphs in Fig. 5. Then $U \subseteq K \subseteq G$, $A(K) \subseteq A(G)$ and $|A(G)|/|A(K)| = 48/2 = 24$. Since G contains 8 triangles and each

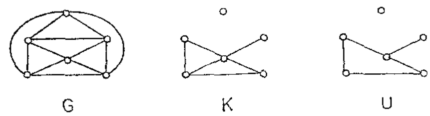


FIGURE 5

triangle is adjacent to three triangles G contains exactly $8 \cdot 3 \cdot 2/2 = 24$ subgraphs isomorphic to K. Hence, $K \in \mathscr{F}(G)$. Again $A(U) = A(K)$ and K contains exactly one subgraph isomorphic to U. Hence $U \in \mathscr{F}(K)$. But $|A(G)|/|A(U)| = 48/2 = 24$ and G contains more than 24 graphs isomorphic to U. Hence $U \notin \mathscr{F}(G)$.

On the other hand we have:

THEOREM 6. *Let $U \subseteq K \subseteq G$, $U \in \mathscr{F}(K)$, $K \in \mathscr{F}(G)$. Then $U \in \mathscr{F}(G)$ if and only if for each subgraph U' of G isomorphic to U there exists a subgraph K' of G such that $U' \subseteq K'$ and K' is isomorphic to K.*

Proof. Let $U \subseteq K \subseteq G$, $U \in \mathscr{F}(K)$, $K \in \mathscr{F}(G)$.

Suppose $U \in \mathscr{F}(G)$. Suppose there exists $U' \subseteq G$, $U' \simeq U$, such that U' is not a subgraph of any subgraph of G which is isomorphic to K. Then, since $U \subseteq K$, U is a subgraph of an element of $\operatorname{int}(G) \backslash \{G\}$. Hence, by Theorem 3, $U \notin \mathscr{F}(G)$ and we have a contradiction.

Now suppose for each $U' \simeq U$ there exists $K' \simeq K$ such that $U' \subseteq K'$. Then, since there exists exactly $|A(G)|/|A(K)|$ subgraphs of G isomorphic to K and exactly $|A(K)|/|A(U)|$ subgraphs of K isomorphic to U, it follows that G contains no more than $|A(G)|\,|A(K)|/|A(K)|\,|A(U)| = |A(G)|/|A(U)|$ subgraphs isomorphic to U. However, since $U \in \mathscr{F}(K)$ and $K \in \mathscr{F}(G)$, from Theorem 1, $A(U) \subseteq A(G)$. Hence G contains at least $|A(G)|/|A(U)|$ subgraphs isomorphic to U. Hence $U \in \mathscr{F}(G)$.

Remark 3. Notice that in the last example $U \subseteq K \subseteq G$, $U \in \mathscr{F}(K)$, $K \in \mathscr{F}(G)$. However, there exists a subgraph U' of G which is not contained in any subgraph of G that is isomorphic to K. Such a subgraph U' is indicated in Fig. 6 by broken lines. Therefore, by Theorem 6, $U \notin \mathscr{F}(G)$.

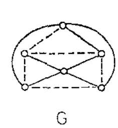


FIGURE 6

2. We consider now some examples of fixing subgraphs of specific graphs. The graphs considered here are very symmetric and contain highly non-trivial fixing subgraphs. These examples have been constructed by inspection using Theorem 1 and Theorem 3. First we illustrate, by a very simple example, the most elementary type of method used:

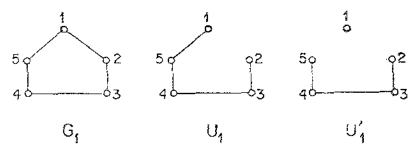


FIGURE 7

EXAMPLE 2. Let G be the pentagon. We consider the labeled graph G_1 in Fig. 7. Delete [1, 2] to obtain the graph U_1. Clearly no edge other than [1, 2] can be adjoined to U_1 to obtain an element of $\pi(G)$. Hence G is the only element of int(G) containing U as a subgraph. Hence, by Theorem 3, $U \in \mathscr{F}(G)$. Next delete [1, 5] from U_1 to obtain the subgraph U_1'. Clearly no two edges other than [1, 2] and [1, 5] can be adjoined to U_1' to obtain an element of $\pi(G)$. Therefore, by Theorem 3, $U' \in \mathscr{F}(G)$. If any edge is deleted from U_1' to obtain a subgraph U_1'', then $A(U'') \notin A(G)$ and, hence, by Theorem 1, $U'' \notin \mathscr{F}(G)$. Thus U' is a minimal fixing subgraph.

We could of course prove $U' \in \mathscr{F}(G)$ directly since $A(U') \subseteq A(G)$ and G contains exactly $|A(G)|/|A(U')| = 10/2 = 5$ subgraphs isomorphic to U'. However, as mentioned above, the definition is usually difficult to apply directly since it is in general very difficult to count the number of subgraphs isomorphic to a given subgraph.

EXAMPLE 3. Let P_n, K_n denote the n-gon and the complete graph on n vertices, respectively. Let K_{mn} denote the complete bipartite graph, i.e., K_{mn} is the complement of the union of the disjoint graphs K_m and K_n. The graph $C_{m,n}$ is defined as follows. If $m = 0$ then $C_{m,n}$ consists of n isolated vertices, and if $m \geqslant 1$ then $C_{m,n}$ consists of $m - 1$ isolated vertices, 1 vertex with degree n, and n vertices with degree 1. Finally we recall, by Theorem 4, all supergraphs of a fixing subgraph are also fixing subgraphs.

(i) $\mathscr{F}(P_n)$ contains the spanning subgraph obtained from P_n by a deletion of an arc of length 2. Furthermore this subgraph is a minimal element of $\mathscr{F}(P_n)$. The proof is obvious by Theorem 3.

(ii) $\mathscr{F}(K_n)$ contains the empty graph Ω_n (i.e., $E(\Omega_n) = \varnothing$). The proof is obvious.

(iii) $\mathscr{F}(K_{mn})$, $m \leqslant n$, contains the graph C_{mn}. To prove this observe that

$$|A(K_{mn})|/|A(C_{mn})| = m!\,n!/(m-1)!\,n! = m, \qquad m \neq n,$$

$$|A(K_{mm})|/|A(C_{mm})| = 2m!^2/(m-1)!\,m! = 2m.$$

Clearly K_{mn} contains m subgraphs isomorphic to C_{mn} when $m \neq n$ and $2m$ subgraphs isomorphic to C_{mn} when $m = n$. Hence, $C_{mn} \in \mathscr{F}(K_{mn})$. Obviously C_{mn} is a minimal element of $\mathscr{F}(K_{mn})$.

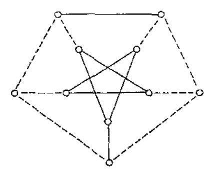


FIGURE 8

(iv) The graph P in Fig. 8 is called the Petersen graph (for the terminology used in this section see [6]). Let U_p be the forest denoted by the broken lines. We now prove $U_p \in \mathscr{F}(P)$. Clearly $|A(U_p)| = 2$. Since P is 3-regular $|A(P)| = 10 \cdot 3 \cdot 2 \cdot 2 = 120$. Hence $|A(P)|/|A(U_p)| = 60$. Fix any vertex of P as the isolated vertex of U_p then, by inspection, there exist just 6 subgraphs of P isomorphic to U_p with this isolated vertex. Since this vertex can be chosen in 10 ways there exist $10 \cdot 6$ subgraphs of P isomorphic to U_p. Obviously U_p is a minimal element of $\mathscr{F}(P)$. The referee points out that the graph in Fig. 9 also belongs to $\mathscr{F}(P)$.

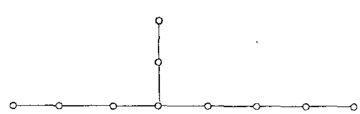


FIGURE 9

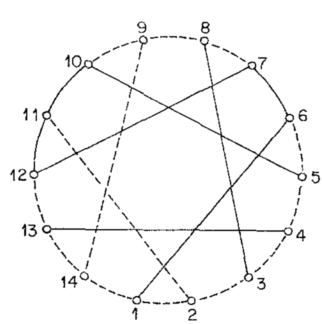


FIGURE 10

(v) The graph H in Fig. 10 is called the Heawood graph. Let U_h be the tree denoted by the broken lines. We prove $U_h \in \mathscr{F}(H)$. Clearly $|A(U_h)| = 1$. Furthermore, since H is 4-regular, $|A(H)| = 14 \cdot 3 \cdot 2^3$.

Therefore $|A(H)|/|A(U_h)| = 14 \cdot 3 \cdot 2^3$. This number is equal to the number of 4-routes in H. Label H as in Fig. 10. Then there exists exactly one subgraph isomorphic to U_h containing the 4-route with initial vertex 2 and terminal vertex 6 such that the degrees of the vertices 2, 3, 4, 5, 6 are respectively 3, 2, 2, 2, 1. Therefore, since H is 4-regular, the number of subgraphs of H isomorphic to U_h is equal to the number of 4-routes in H. Therefore H contains exactly $|A(H)|/|A(U_h)|$ subgraphs isomorphic to U_h. Hence, $U_h \in \mathscr{F}(H)$. We do not know if U_h is minimal or not. The referee points out that the graphs in Fig. 11 also belong to $\mathscr{F}(H)$.

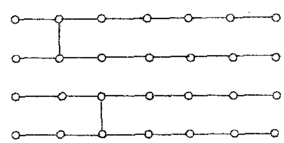


FIGURE 11

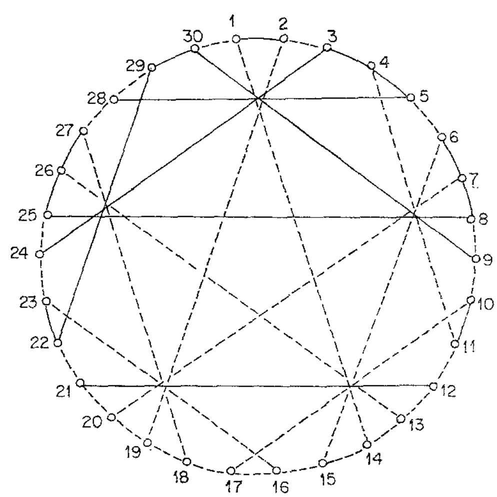


FIGURE 12

The graph L in Fig. 12 is the 8-cage, known to geometers as the Levi graph of the Cremona-Richmond Configuration. The tree U_l, denoted by the broken lines in Fig. 12, is a fixing subgraph. We now prove this remark. Select the 5-arc $s = 18, 17, 16, 15, 14, 13$. Let this be the 5-arc in U_l which has each of its vertices of degree 3 in U_l. Furthermore suppose the vertex labeled 18 is adjacent to a vertex of degree 3 in U_l. Then s uniquely determines (see [10]) the subgraph U_l (this is fairly easy to check).

Hence, since L is 5-regular, every subgraph of L isomorphic to U_l is also similar to U_l. Since $|A(U_l)| = 1$ and $|A(L)| = 1440$, it follows that L contains exactly 1440 subgraphs isomorphic to U_l. Hence, $U_l \in \mathscr{F}(L)$.

It follows from these examples that:

THEOREM 7. *If G is one of Tutte's cages* [7] *there exists a forest $F \in \mathscr{F}(G)$.*

Remark 4. It seems likely that, if G is a regular graph, then its fixing graphs will be very non-trivial. It is hoped that this theorem will emphasize the significance of fixing graphs in this context.

3. We consider in this and the following section an extremal problem. Let G be a graph. Since G contains exactly $|A(G)|/|A(G)| = 1$ subgraph isomorphic to G, we deduce that $G \in \mathscr{F}(G)$.

$$\text{“Which graphs } G\text{, if any, satisfy } |\mathscr{F}(G)| = 1\text{?”} \qquad \text{(E. 1)}$$

Theorem 8 below establishes the existence of such graphs. Let $\mathscr{D}(G)$ denote the set of spanning subgraphs obtained by deleting one edge of G in all possible ways.

PROPOSITION 1. *Let G be a graph. A necessary condition for $|\mathscr{F}(G)| > 1$ is that G is stable.*

Proof. Suppose $|\mathscr{F}(G)| > 1$. Then, from Theorem 4,

$$\mathscr{F}(G) \cap \mathscr{D}(G) \neq \varnothing.$$

Therefore there exists $U \in \mathscr{F}(G) \cap \mathscr{D}(G)$. By definition there exists $\lambda \in E(G)$ such that $U \equiv G - \lambda$. Since $G - \lambda \in \mathscr{F}(G)$, by Theorem 1, $A(G - \lambda) \subseteq A(G)$. Hence G is stable.

THEOREM 8. *Let $\tilde{K}_n \in \mathscr{D}(K_n)$ then $|\mathscr{F}(\tilde{K}_n)| = 1, n > 1$.*

Proof. If $U \in \mathscr{D}(\tilde{K}_n)$, $A(U) \not\subseteq A(\tilde{K}_n)$. Hence, for all $\lambda \in E(\tilde{K}_n)$, $A(\tilde{K}_n - \lambda) \not\subseteq A(\tilde{K}_n)$. Therefore $\tilde{K}_n$ is unstable. Therefore, by Proposition 1, $|\mathscr{F}(\tilde{K}_n)| = 1$.

We now prove that, if T is a non-trivial tree, then $|\mathscr{F}(T)| = 1$ if and only if T is an arc of length $\geqslant 2$ or is isomorphic to T_1 or T_2 (see Fig. 1). It turns out that the necessary condition for $|\mathscr{F}(T)| > 1$ given in Proposition 1 is also sufficient.

Preliminaries. Let G be a finite graph. G is *trivial* if $|V(G)| \leqslant 1$. Let $\xi \in V(G)$ then $d(\xi)$ denotes the degree of ξ. Let $A_\xi(G) = \{\sigma \in A(G)\colon \xi\sigma = \xi\}$.

An edge of G is *pendant* if at least one end-vertex of the edge has degree 1. Let $\mathscr{P}(G) = \{\lambda \in E(G): \lambda \text{ is pendant}\}$. Let $\mathscr{P}_\xi(G)$ be the subset of $\mathscr{P}(G)$ consisting of those edges having ξ as an end-vertex.

Let s be a finite sequence. The first and last terms of s are denoted, respectively, by $F(s)$ and $L(s)$. If s_1, s_2 are sequences such that $L(s_1) = F(s_1)$, then s_1s_2 denotes the sequence obtained by writing down all the terms of s_1 followed by all the terms of s_2 other than the first. An *arc* π in G is a sequence

$$\xi_0, \xi_1, \xi_2, ..., \xi_n \ (n \geqslant 0),$$

where $\xi_i \in V(G)$, $\xi_i \neq \xi_j$ when $i \neq j$ and $[\xi_i, \xi_{i+1}] \in E(G), i \in \{0, 1, 2, ..., n-1\}$. The *length* of π is n and is denoted by $\ell(\pi)$. A *subarc* of π is a subsequence of consecutive terms of π. A *circuit* in G is a sequence π as above except that $\xi_0 = \xi_n$ and $n \geqslant 1$. A *subarc of* a circuit π is a subarc of π (or any rotation of π).

Let $\lambda \in \mathscr{P}(T)$. Suppose $\lambda = [\xi, \eta]$ where ξ is an end-vertex of λ of degree 1. Let $T - \lambda$ denote the subgraph of T defined by

$$V(T - \lambda) = V(T) - \{\xi\}, E(T - \lambda) = E(T) - \{\lambda\}.$$

T is *$\mathscr{P}(T)$-stable* if there exists $\lambda \in \mathscr{P}(T)$ such that $A(T - \lambda) \subseteq A(T)$ and T is *$\mathscr{P}(T)$-unstable* otherwise. Notice that, if $A(T - \lambda) \nsubseteq A(T)$, then there exists $\sigma \in A(T - \lambda)$ such that $\eta\sigma \neq \eta$.

THEOREM 9 (see [8]). *Let T be a non-trivial $\mathscr{P}(T)$-unstable tree. Then T is an arc of length $\geqslant 2$ or T is isomorphic to T_1 or T_2 (see Fig. 1).*

LEMMA (see [3, Theorem 4]). *Let T be a tree. Let $\lambda, \mu \in \mathscr{P}(T)$. If $T - \lambda$ and $T - \mu$ are isomorphic then they are similar.*

THEOREM 10. *Let T be a tree. Then $|\mathscr{F}(T)| = 1$ if and only if T is a non-trivial arc or T is isomorphic to T_1 or T_2.*

Proof. Let T be a tree. Suppose T is neither an arc nor isomorphic to T_1 or T_2. From Theorem 9, T is $\mathscr{P}(T)$-stable i.e., there exists $\lambda \in \mathscr{P}(T)$ such that $A(T - \lambda) \subseteq A(T)$. Suppose T contains a spanning subgraph T' such that T' is isomorphic to $T - \lambda$ then, since $\lambda \in \mathscr{P}(T)$, there exists $\mu \in \mathscr{P}(T)$ such that $T' \equiv T - \mu$. By the lemma, $T - \lambda$ and T' are similar. Hence, since $A(T - \lambda) \subseteq A(T)$, T contains exactly $|A(T)|/|A(T - \lambda)|$ subgraphs isomorphic to $T - \lambda$. Therefore, $T - \lambda \in \mathscr{F}(T)$. Therefore, since $T \in \mathscr{F}(T)$, $|\mathscr{F}(T)| > 1$.

Suppose T is a non-trivial arc or T is isomorphic to T_1 or T_2. Then, from Theorem 9, T is unstable. Hence from Proposition 1, $|\mathscr{F}(T)| = 1$.

4. In this section we prove (Theorem 11) the analog of Theorem 9 for monocyclic graphs. We prove in fact that, if M is an unstable monocyclic graph, then M is isomorphic to one of the graphs in Fig. 2. Hence for any monocyclic graph M which is not isomorphic to one of these graphs there exists $U \in \mathscr{D}(M)$ such that $A(U) \subseteq A(M)$. However the analog of the lemma of Section 3 is false, i.e., if $\lambda, \mu \in \mathscr{P}(M)$ and if $M - \lambda$ and $M - \mu$ are isomorphic, then $M - \lambda$ and $M - \mu$ are not necessarily similar. As a consequence the condition (see Proposition 1) that M is stable is necessary but not sufficient for $|\mathscr{F}(M)| > 1$.

Preliminaries. A finite graph is *monocyclic* if it is connected and contains exactly one circuit. A monocyclic graph M with circuit π will sometimes be denoted by (M, π). A monocyclic graph M is *admissible* if it is $\mathscr{P}(M)$-unstable. Let (M, π) be a monocyclic graph where

$$\pi \equiv \xi_0, \xi_1, \xi_2, \ldots, \xi_n .$$

Let $T_1, T_2, \ldots, T_n$ denote the sequence of maximal subtrees of M satisfying $V(T_i) \cap V(\pi) = \{\xi_i\}$, $i = 1, 2, \ldots, n$. If T_i is trivial then ξ_i is *branch-free*. Since M is monocyclic we have:

$$V(T_i) \cap V(T_j) = \varnothing, \qquad i \neq j; \qquad G = \pi \cup T_1 \cup T_2 \cup \cdots \cup T_n . \quad (1)$$

If $\xi \in V(\pi)$, $\xi \equiv \xi_i$ (say), it will sometimes be notationally convenient to write T_ξ instead of T_i. If T_i is an arc then ξ_i is *arc-like* and we write $d^+(\xi_i) \equiv l(T_i) + 1$. Hence if ξ_i is branch-free $d^+(\xi_i) = 1$.

Let s be a subarc of π. If each element of $V(s)$ is branch-free then s is *branch-free*. Let $\omega(M, \pi) \equiv \max\{\ell(s) : s \text{ is a branch-free subarc of } \pi\}$. If s is branch-free and $\ell(s) = \omega(M, \pi)$ then s is *maximum branch-free*. Let $\Delta(M, \pi) \equiv \{s : s \text{ is a maximal branch-free subarc of } \pi\}$. Finally if s is a subarc of π let $F_-(s)$ denote the vertex of π which precedes $F(s)$ in π and let $L_+(s)$ denote the vertex of π which succeeds $L(s)$ in π.

LEMMA 1. *Let (M, π) be an admissible graph. Let $\xi \in V(\pi)$. If ξ is not branch-free there exist $\lambda \in \mathscr{P}(T_\xi) \cap \mathscr{P}(M)$ and $\sigma_\lambda \in A(M - \lambda)$ such that $\xi\sigma_\lambda \neq \xi$.*

Proof. Suppose, for all $\lambda \in \mathscr{P}(T_\xi) \cap \mathscr{P}(M)$ and $\sigma_\lambda \in A(M - \lambda)$, $\xi\sigma_\lambda = \xi$. Since M is $\mathscr{P}(M)$-unstable we deduce that T_ξ is $\mathscr{P}(M) \cap \mathscr{P}(T_\xi)$-unstable. We obtain a contradiction immediately as a consequence of Theorem 9.

LEMMA 2. *Let (M, π) be an admissible graph. There exist $\eta \in V(\pi)$ such that η is branch-free.*

Proof. Let (M, π) be an admissible graph. If (M, π) is a polygon then the lemma is true. Therefore suppose (M, π) is not a polygon. Choose $\xi \in V(\pi)$ subject to:

(i) ξ is not branch-free;

(ii) $|V(T_\xi)|$ is minimal consistent with (i).

By Lemma 1, there exist $\lambda \in \mathscr{P}(T_\xi) \cap \mathscr{P}(M)$ and $\sigma_\lambda \in A(M - \lambda)$ such that $\xi\sigma_\lambda = \eta, \eta \neq \xi$. Clearly $\eta \in V(\pi)$ and $|V(T_\eta)| = |V(T_\xi)| - 1$, i.e., $|V(T_\eta)| < |V(T_\xi)|$. Therefore, by the choice of ξ, η is branch-free.

Notation 1. For Lemmas 3, 4, 5, and 6 we require the following notation. Let (M, π) be an admissible graph. Let $\pi \equiv \xi_0, \xi_1, ..., \xi_n$. Let s be a branch-free subarc of π (by Lemma 2 such an arc exists). Let

$$\xi_\alpha \equiv F_-(s), \qquad \xi_\beta \equiv L_+(s).$$

Suppose $\alpha \leqslant \beta$ (otherwise relabel).

LEMMA 3. *Let (M, π) be an admissible graph. Suppose $\omega(M, \pi) < \ell(\pi) - 2$. There exists $s \in \Delta(M, \pi)$ such that ξ_α, ξ_β are arc-like and either*

(i) $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$

or

(ii) $\{d^+(\xi_\alpha), d^+(\xi_\beta)\} = \{2, 3\}$.

Proof. Let (M, π) be an admissible graph. By Lemma 2, $\Delta(M, \pi) \neq \varnothing$. Choose $s \in \Delta(M, \pi)$ so that $\delta(s) \equiv |V(T_\alpha)| + |V(T_\beta)|$ is as small as possible ($\xi_\alpha \neq \xi_\beta$ since $\omega(M, \pi) < \ell(\pi) - 2$). We assume $|V(T_\alpha)| \leqslant |V(T_\beta)|$ (otherwise relabel).

Since $s \in \Delta(M, \pi)$, ξ_α is not branch-free and therefore, by Lemma 1, there exist $\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$ and $\sigma_\alpha \in A(M - \lambda_\alpha)$ such that

$$\xi_\alpha\sigma_\alpha = \eta_\alpha \neq \xi_\alpha, \qquad \eta_\alpha \in V(\pi). \tag{1}$$

If $\xi_\alpha \notin V(\xi_\alpha\sigma_\alpha(s\sigma_\alpha)\,\xi_\beta\sigma_\alpha)$ then $s\sigma_\alpha \in \Delta(M, \pi)$ and $\delta(s\sigma_\alpha) < \delta(s)$, which is a contradiction.

Therefore $\xi_\alpha \in V(\xi_\alpha\sigma_\alpha(s\sigma_\alpha)\,\xi_\beta\sigma_\alpha)$. Since $|V(T_\alpha)| \leqslant |V(T_\beta)|$, $\xi_\alpha \neq \xi_\beta\sigma_\alpha$. Therefore $\xi_\alpha \in V(s\sigma_\alpha)$. Therefore ξ_α is branch-free in $M - \lambda_\alpha$. Therefore ξ_α

is arc-like in M and $d^+(\xi_\alpha) = 2$. By Lemma 1 there exist $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$ and $\sigma_\beta \in A(M - \lambda_\beta)$ such that

$$\xi_\beta \sigma_\beta = \eta_\beta \neq \xi_\beta , \qquad \eta_\beta \in V(\pi). \tag{2}$$

If $\xi_\beta \notin V(\xi_\alpha \sigma_\beta (s\sigma_\beta)\, \xi_\beta \sigma_\beta)$ then $s\sigma_\beta \in \Delta(M, \pi)$ and $\delta(s\sigma_\beta) < \delta(s)$, which is a contradiction. Therefore $\xi_\beta \in V(\xi_\alpha \sigma_\beta (s\sigma_\beta)\, \xi_\beta \sigma_\beta)$. Therefore $\xi_\beta \in V(\xi_\alpha \sigma_\beta (s\sigma_\beta))$. If $\xi_\beta \in V(s\sigma_\beta)$ then ξ_β is branch-free in $M - \lambda_\beta$. Therefore ξ_β is arc-like in M and $d^+(\xi_\beta) = 2$. In this case $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$. Suppose $\xi_\beta \notin V(s\sigma_\beta)$, then $\xi_\beta = \xi_\alpha \sigma_\beta$. Therefore ξ_β is arc-like in $M - \lambda_\beta$ and $d^+_{M-\lambda_\beta}(\xi_\beta) = 2$. Therefore ξ_β is arc-like in M and $d^+(\xi_\beta) = 3$. In this case $d^+(\xi_\alpha) = 2$, $d^+(\xi_\beta) = 3$.

Notation 2. Let (M, π) be an admissible graph. Suppose:

(T. 1) $V(\pi)$ contains exactly two vertices ξ_α, ξ_β which are not branch-free; in addition ξ_α, ξ_β are arc-like and $\{d^+(\xi_\alpha), d^+(\xi_\beta)\} = \{2, 3\}$ or $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$;

or

(T. 2) $V(\pi)$ contains exactly three vertices ξ_α, ξ_β, ξ_γ which are not branch-free; in addition ξ_α, ξ_β, ξ_γ are arc-like and $d^+(\xi_\alpha) = d^+(\xi_\beta) = d^+(\xi_\gamma) = 2$;

or

(T.3) after relabeling the elements of $V(\pi)$ if necessary, there exists a positive integer k such that $\xi_\alpha \in V(\pi)$ is branch-free unless

$$\alpha \in \left\{k + r\,\frac{(k+1)}{2} : r = 0, 1,\dots, 2\,\frac{(n-k-1)}{k+1}\right\},$$

in which case $d^+(\xi_\alpha) = 2$, i.e., (M, π) is the graph in Fig. 13 when $k = 3$, $n = 12$; then (M, π) is *almost polygonal*, e.g., all the graphs in Fig. 2 are almost polygonal graphs.

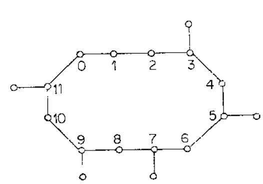


FIGURE 13

LEMMA 4. *Let (M, π) be an admissible graph. Suppose $\omega(M, \pi) < \ell(\pi) - 2$.*

Let $s \in \Delta(M, \pi)$ and suppose $s \equiv b_1, b_2, \ldots, b_m$. If possible choose s so that ξ_α, ξ_β are arc-like and $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$. Then there exist p, $q \in \{1, 2, \ldots, m\}$, $\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$, $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$, $\sigma_\alpha \in A(M - \lambda_\alpha)$, $\sigma_\beta \in A(M - \lambda_\beta)$ such that

$$\xi_\alpha\sigma_\alpha = b_p\,,\ \xi_\beta\sigma_\beta = b_q\,,\ \xi_\alpha\sigma_\alpha{}^2 = \xi_\alpha\,,\ \xi_\beta\sigma_\beta{}^2 = \xi_\beta\,.$$

If no such subarc of π exists, s may be chosen so that ξ_α, ξ_β are arc-like and $d^+(\xi_\alpha) = 2, d^+(\xi_\beta) = 3$. Furthermore, in this case, there exist $p \in \{1, 2, \ldots, m\}$, $\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$, $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$, $\sigma_\alpha \in A(M - \lambda_\alpha)$, $\sigma_\beta \in A(M - \lambda_\beta)$ such that

$$\xi_\alpha\sigma_\alpha = b_p\,,\ \xi_\beta\sigma_\beta = \xi_\alpha\,,\ \xi_\alpha\sigma_\alpha{}^2 = \xi_\alpha\,,\ \xi_\beta\sigma_\beta{}^2 = \xi_\beta\,.$$

Proof. (i) Let $s \in \Delta(M, \pi)$. Suppose ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$. Since $\omega(M, \pi) < \ell(\pi) - 2$, $\xi_\alpha \neq \xi_\beta$. By Lemma 1 there exist $\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$, $\sigma_\alpha \in A(M - \lambda_\alpha)$ such that $\xi_\alpha\sigma_\alpha \neq \xi_\alpha$. If $\xi_\alpha \notin V(\xi_\alpha\sigma_\alpha(s\sigma_\alpha)\,\xi_\beta\sigma_\alpha)$ then $\xi_\alpha\sigma_\alpha(s\sigma_\alpha)$ is a branch-free subarc of π contained in M. However $\ell(\xi_\alpha\sigma_\alpha(s\sigma_\alpha)) > \ell(s)$, which is a contradiction. Therefore $\xi_\alpha \in V(\xi_\alpha\sigma_\alpha(s\sigma_\alpha)\,\xi_\beta\sigma_\alpha)$, i.e., $\xi_\alpha \in V(s\sigma_\alpha)$. Hence σ_α does not induce a rotation of π, otherwise $\xi_\beta\sigma_\alpha \in V(s)$. Therefore σ_α induces a reflection of π. Hence $\xi_\alpha\sigma_\alpha \in V(s\sigma_\alpha{}^2) = V(s)$. Therefore there exists $p \in \{1, 2, \ldots, m\}$ such that $\xi_\alpha\sigma_\alpha = b_p$, $\xi_\alpha\sigma_\alpha{}^2 = \xi_\alpha$. Similarly there exist $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$, $\sigma_\beta \in A(M - \lambda_\beta)$, $q \in \{1, 2, \ldots, m\}$, such that $\xi_\beta\sigma_\beta = b_p$ where $\xi_\beta\sigma_\beta{}^2 = \xi_\beta$.

(ii) Suppose $\Delta(M, \pi)$ contains no element s such that ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$. Then, by Lemma 3, we may choose s so that ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = 2$, $d^+(\xi_\beta) = 3$. As in (i) above there exist $\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$, $\sigma_\alpha \in A(M - \lambda_\alpha)$, $p \in \{1, 2, \ldots, m\}$, such that $\xi_\alpha\sigma_\alpha = b_p$, $\xi_\alpha\sigma_\alpha{}^2 = \xi_\alpha$. By Lemma 1 there exist $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$, $\sigma_\beta \in A(M - \lambda_\beta)$, such that $\xi_\beta\sigma_\beta \neq \xi_\beta$. If $\xi_\beta \notin V(\xi_\alpha\sigma_\beta(s\sigma_\beta)\,\xi_\beta\sigma_\beta)$ then $s\sigma_\beta \in \Delta(M, \pi)$, $\xi_\alpha\sigma_\beta$, $\xi_\beta\sigma_\beta$ are arc-like in M and $d^+(\xi_\alpha\sigma_\beta) = d^+(\xi_\beta\sigma_\beta) = 2$. By assumption, no such element of $\Delta(M, \pi)$ exists. Therefore $\xi_\beta \in V(\xi_\alpha\sigma_\beta(s\sigma_\beta)\,\xi_\beta\sigma_\beta)$, i.e., $\xi_\alpha\sigma_\beta = \xi_\beta$. Hence σ_β does not induce a rotation of π, otherwise $\Delta(M, \pi)$ contains an element satisfying the conditions in Case (i) above. Therefore σ_β induces a reflection of π and $\xi_\beta\sigma_\beta = \xi_\alpha\sigma_\beta{}^2 = \xi_\alpha$.

LEMMA 5. *Let (M, π) be an admissible graph. Suppose $\omega(M, \pi) < \ell(\pi) - 2$. Then (M, π) is almost polygonal.*

Proof. In the proof we use the notation of Lemma 4. Let (M, π) be an admissible graph. Suppose $\omega(M, \pi) < \ell(\pi) - 2$.

(i) Let $s \in \Delta(M, \pi)$. Suppose ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = d^+(\xi_\beta) = 2$. Then, by Lemma 4, there exist p, $q \in \{1, 2, \ldots, m\}$,

$\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$, $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$, $\sigma_\alpha \in A(M - \lambda_\alpha)$, $\sigma_\beta \in A(M - \lambda_\beta)$, such that

$$\xi_\alpha \sigma_\alpha = b_p\,, \qquad \xi_\beta \sigma_\beta = b_q\,, \qquad \xi_\alpha {\sigma_\alpha}^2 = \xi_\alpha\,, \qquad \xi_\beta {\sigma_\beta}^2 = \xi_\beta\,. \tag{1}$$

We may now assume $\xi_\alpha \equiv F(\pi)$, otherwise relabel. By observing that σ_α, σ_β induce reflections of π we deduce from (1) that, if $\xi_\beta \sigma_\alpha = \xi_\beta$, then $\pi \equiv \xi_\alpha s \xi_\beta s_1 \xi_\alpha$, where s_1 is a branch-free subarc of π. In this case (M, π) is almost-polygonal of type T. 1. Assume therefore that $\xi_\beta \sigma_\alpha \neq \xi_\beta$. Then, again by observing that σ_α, σ_β induce reflections of π, we deduce from (1) that there exists a positive integer k such that

$$\pi \equiv \xi_\alpha s \xi_\beta s_1 \eta_1 s_2 \eta_2 s_3 \cdots \eta_{2k-1} s_{2k} \xi_\alpha\,, \tag{2}$$

where η_i is arc-like, $d^+(\eta_i) = 2$, s_i is branch-free and

$$\ell(s_1) = \ell(s_3) = \cdots = \ell(s_{2k-1}) = q - 2,$$

$$\ell(s_2) = \ell(s_4) = \cdots = \ell(s_{2k}) = m - p - 1.$$

Assume $k > 1$. By Lemma 1, there exist $\lambda_1 \in \mathscr{P}(T_{\eta_1}) \cap \mathscr{P}(M)$, $\sigma_1 \in A(M - \lambda_1)$ such that $\eta_1 \sigma_1 \neq \eta_1$. If $\eta_1 \notin V((s_1\sigma_1)\, \eta_1\sigma_1(s_2\sigma_1))$, then $(s_1\sigma_1)\, \eta_1\sigma_1(s_2\sigma_1)$ is a branch-free subarc of π contained in M and $\ell((s_1\sigma_1)\, \eta_1\sigma_1(s_2\sigma_1)) > \ell(s_i)$, $i \in \{1, 2,\ldots, 2k\}$. Therefore $(s_1\sigma_1)\, \eta_1\sigma_1(s_2\sigma_1) \equiv s$ (or s^{-1}). Clearly, since $\ell(s) > \ell(s_3)$, $\ell(s) > \ell(s_{2k})$, σ_1 does not induce a rotation of π. Hence $\eta_2\sigma_1 = \xi_\alpha$, $\xi_\beta\sigma_1 = \xi_\beta$. Therefore $\ell(s_3) = \ell(s_{2k})$. Therefore it is clear from the symmetry of (2) that (M, π) is almost polygonal of type T.3. In the case $k = 1$, from (2), (M, π) is almost polygonal of type T.2. On the other hand suppose $\eta_1 \in V((s_1\sigma_1)(\eta_1\sigma_1)(s_2\sigma_1))$ (and $k > 1$) then, since $\ell(s) > \ell(s_{2k})$, σ_1 does not induce a rotation of π. Therefore $\xi_\beta\sigma_1 = \eta_2$. Therefore $s\sigma_1 = s_3$. Therefore $\ell(s) = \ell(s_3)$, which is impossible. Therefore $k = 1$ and again (M, π) is almost polygonal of type T.1.

(ii) Suppose $\Delta(M, \pi)$ contains no element s such that ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = d^+(\xi_\beta) = 1$. Then, by Lemma 3, we may choose $s \in \Delta(M, \pi)$ such that ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = 2$, $d^+(\xi_\beta) = 3$. By Lemma 4, there exist $p \in \{1, 2,\ldots, m\}$,

$\lambda_\alpha \in \mathscr{P}(T_\alpha) \cap \mathscr{P}(M)$, $\lambda_\beta \in \mathscr{P}(T_\beta) \cap \mathscr{P}(M)$, $\sigma_\alpha \in A(M - \lambda_\alpha)$, $\sigma_\beta \in A(M - \lambda_\beta)$

such that

$$\xi_\alpha \sigma_\alpha = b_p\,, \qquad \xi_\beta \sigma_\beta = \xi_\alpha\,, \qquad \xi_\alpha {\sigma_\alpha}^2 = \xi_\alpha\,, \qquad \xi_\beta {\sigma_\beta}^2 = \xi_\beta\,. \tag{3}$$

By observing that σ_α, σ_β induce reflections of π we deduce from (3) that, if $\xi_\beta\sigma_\alpha = \xi_\beta$, then $\pi \equiv \xi_\alpha s \xi_\beta s_1 \xi_\alpha$, where s_1 is a branch-free subarc of π. In this case (M, π) is almost polygonal of type T.1. Assume therefore that $\xi_\beta\sigma_\alpha \neq \xi_\beta$. Then, again by observing that σ_α, σ_β induce reflections of π, we deduce from (3) that there exists a positive integer k such that

$$\pi \equiv \xi_\alpha s \xi_\beta s_1 \eta_1 s_2 \eta_2 s_3 \eta_3 \cdots \eta_k s_{k+1} \xi_\alpha, \tag{4}$$

where η_i is arc-like, $d^+(\eta_i) = 3$, s_i is branch-free, and

$$\ell(s_1) = \ell(s_2) = \cdots = \ell(s_k) = m - p - 1.$$

By Lemma 1, there exist $\lambda_1 \in \mathscr{P}(T_{\eta_1}) \cap \mathscr{P}(M)$, $\sigma_1 \in A(M - \lambda_1)$, such that $\eta_1\sigma_1 \neq \eta_1$. Since $d^+(\eta_1) = 3$, $d^+(\eta_1\sigma_1) = 2$. Therefore, from (4), $\eta_1\sigma_1 = \xi_\alpha$. Therefore, from (4), $(s_1\sigma_1)\, \eta_1\sigma_1(s_2\sigma_1)$ is $s_{k+1}\xi_\alpha s$ or its inverse. Therefore $(s_1\sigma_1)(\eta_1\sigma_1)(s_2\sigma_1)$ and $s_{k+1}\xi_\alpha s$ have the same length. Hence $m - p - 1 + 1 + m - p - 1 = m - p - 1 + 1 + m - 1$. Hence $p = 0$, which is a contradiction. This completes the proof of the lemma.

LEMMA 6. *Let (M, π) be an unstable monocyclic graph. Suppose $\omega(M, \pi) < \ell(\pi) - 2$. Then $\omega(M, \pi) \leqslant 3$.*

Proof. Let (M, π) be an unstable monocyclic graph. Then (M, π) is $\mathscr{P}(M)$-unstable and so (M, π) is an admissible graph. Suppose $\omega(M, \pi) < \ell(\pi) - 2$. By Lemma 5, (M, π) is almost polygonal. Let $s \in \Delta(M, \pi)$ and suppose $s \equiv b_1, b_2, \ldots, b_m$.

(i) Suppose ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = 2$, $d^+(\xi_\beta) = 3$. Assume $m > 3$ and let $\lambda = [b_{m-1}, b_m]$. Then $A(M - \lambda) \subseteq A(M)$. Hence (M, π) is stable, which is a contradiction. Therefore $m \leqslant 3$.

(ii) Suppose ξ_α, ξ_β are both arc-like and $d^+(\xi_\alpha) = 2$, $d^+(\xi_\beta) = 2$. Assume $m > 3$ and let $\lambda = [b_i, b_{i+1}]$, $2 \leqslant i \leqslant m - 2$. Then $A(M - \lambda) \subseteq A(M)$ and again we obtain a contradiction. Hence $m \leqslant 3$.

Since $\Delta(M, \pi)$, by the definition of almost polygonal, contains an element s satisfying (i) or (ii), the lemma is proved.

Remark 5. It is now very easy to determine all admissible (or $\mathscr{P}(M)$-unstable) graphs M.

THEOREM 11. *Let (M, π) be an unstable monocyclic graph. Then (M, π) is isomorphic to one of the graphs in Fig. 2.*

Proof. Let (M, π) be an unstable monocyclic graph. Suppose

$\omega(M, \pi) = \ell(\pi)$, then (M, π) is a polygon and therefore stable, which is a contradiction. Clearly $\omega(M, \pi) \neq \ell(\pi) - 1$. Suppose $\omega(M, \pi) = \ell(\pi) - 2$. Then $V(\pi)$ contains just one element ξ which is not branch-free. By Lemma 1, there exist $\lambda \in \mathscr{P}(T_\xi) \cap \mathscr{P}(M)$, $\sigma_\lambda \in A(M - \lambda)$ such that $\xi\sigma_\lambda \neq \xi$. Since $\xi\sigma_\lambda \in V(\pi)$, $\xi\sigma_\lambda$ is branch-free. Hence ξ is arc-like and $d^+(\xi) = 2$. By Lemma 6, $\ell(\pi) = \omega(M, \pi) + 1 \leqslant 4$. Hence M is isomorphic to M_1 or M_4.

We may therefore assume $\omega(M, \pi) < \ell(\pi) - 2$. By Lemma 5, (M, π) is almost polygonal and, by Lemma 6, $\omega(M, \pi) \leqslant 3$. It is a very simple exercise now to check that (M, π) is isomorphic to M_2, M_3, M_5, or M_6 if (M, π) is almost polygonal of type (T. 1) or (T. 2). If (M, π) is almost polygonal of type (T. 3) it is also easy to check that (M, π) is isomorphic to one of the infinite family of graphs M_5, M_6, M_7,... or to M_4.

Remark 6. Let M be an unstable monocyclic graph. Then, by Proposition 2, $|\mathscr{F}(M)| = 1$. Therefore, by Theorem 11, $|\mathscr{F}(M)| = 1$ if M is isomorphic to one of the graphs in Fig. 2.

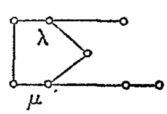


FIGURE 14

Let M be a monocyclic graph then from Proposition 1, if $|\mathscr{F}(M)| > 1$ M is stable. However this is only a necessary condition for $|\mathscr{F}(M) > 1$. The graph M_0 (Fig. 14) is stable but $|\mathscr{F}(M_0)| = 1$. Hence the stability of M is not a sufficient condition for $|\mathscr{F}(M)| > 1$. This is because the analog of the main lemma of Section 3 for monocyclic graphs is false. Thus $M_0 - \lambda$ and $M_0 - \mu$ are isomorphic but not similar graphs. The basic reason why the analog of the lemma to Theorem 10 is not true is that for any two *similar vertices of a tree there is an automorphism which interchanges* them. This is not true for monocyclic graphs. We have not been able to obtain sufficient conditions for $|\mathscr{F}(M)| > 1$.

5. We may consider an extremal problem related to the extremal problem (E. 1) posed in Section 3:

$$\text{“Which graphs } G\text{, if any, satisfy } \mathscr{D}(G) \subseteq \mathscr{F}(G)\text{?”} \qquad \text{(E.2)}$$

Theorem 12 below establishes the existence of such graphs.

PROPOSITION 2. *Let G be a graph. A necessary condition for*

$$\mathscr{D}(G) \subseteq \mathscr{F}(G)$$

is that G is completely stable.

Proof. Let $\lambda \in E(G)$. Then $G - \lambda \in \mathscr{D}(G)$. Therefore, $A(G - \lambda) \subseteq A(G)$.

THEOREM 12. *Let G be a regular graph (i.e., all the vertices of G have the same degree). Then* $\mathscr{D}(G) \subseteq \mathscr{F}(G)$.

Proof. Follows immediately from Theorem 3.

Remarks. 1°. To obtain sufficient conditions either for $\mathscr{D}(G) \subseteq \mathscr{F}(G)$ or for $|\mathscr{F}(G)| > 1$ seems in general to be very difficult.

2°. From Proposition 1, if G is a graph and $|\mathscr{F}(G)| > 1$ then G is stable. It would be interesting to characterize those graphs which are stable.

Comment. Some slight information on this problem is given by:

(1) From Theorems 9 and 11 monocyclic graphs and trees are, with a few trivial exceptions, stable,

(2) We deduce from P. Erdös and A. Rényi [1] that for large values of n most (in its mathematical sense) graphs are completely stable.

3°. From Proposition 2, if G is a graph and $\mathscr{D}(G) \subseteq \mathscr{F}(G)$ then G is completely stable, e.g., when G is a regular graph. Completely stable graphs have been discussed in [1] and [5].

4°. Finally we notice that in [4] it is stated that, in general, there is no relationship between the automorphism group of a graph and the automorphism group of a subgraph. By imposing some conditions on the occurrence of the isomorphic copies of a subgraph in the graph it is hoped that this paper has shown some relationship between the automorphism group of a graph and the automorphism group of a subgraph.

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I am deeply indebted to the referee for his very valuable comments. Among other important suggestions he informs me that Theorem 11 is not sufficiently strongly made. In fact he has proved that: "A connected monocyclic graph M with $|\mathscr{F}(M)| = 1$ is isomorphic to one of the graphs in Fig. 2 or Fig. 14"; i.e., with the exception of the graph in Fig. 14 the stability of M is a sufficient condition for $|\mathscr{F}(M)| > 1$.

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